Risk-Based Portfolio Optimization Using SAS®

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ABSTRACT

In the midst of the recent turbulence in financial markets, risk management has become an increasingly critical part of the decision-making process in financial institutions. A risk-based profitability strategy helps increase firm value and helps avoid costs associated with financial distress. This paper explains how advanced risk analysis and optimization products from SAS® can work together to enable optimal, risk-adjusted portfolio decisions. Several portfolio optimization problems are discussed: (1) maximization of portfolio return subject to a target risk tolerance, expressed as standard risk measures such as value at risk (VaR); (2) minimization of asset/liability mismatch through cash flow allocation minimization; and (3) minimization of capital requirements by employment of an optimal risk mitigation strategy. For these problems, best-practice solutions are presented using SAS Risk Management solutions.

INTRODUCTION

Modern financial theory has supported a strong relationship between profitability and risk. Risk borne by an investment should be compensated by excess return. Financial institutions such as banks are known to take the risk intermediation function for the entire economy. Adequate recognition of the underlying risk of an investment strategy is a critical part of a financial decision process. Markowitz (1952) introduced the seminal risk-based portfolio optimization model, which won him the 1990 Nobel Prize for Economics, together with William Sharpe and Merton Miller. (All three laureates were awarded the Nobel Prize for their work in risk-based decision theories in investment and corporate finance.)

The past few decades have seen tremendous advances in financial markets. The innovations of financial products and financial management tools have contributed many good aspects to economic development. The complication in the financial market has made both researchers and practitioners realize the limitation of the normality assumption in the earlier work of the risk portfolio theory. In the early 1990s, J. P. Morgan proposed a new risk measure, value at risk (VaR), which is a possible loss or value at a certain confidence interval. Statistically, VaR is simply a percentile of a distribution. It extends the standard deviation to a measure that does not restrict its application to the normal distribution. Other risk measures have been proposed as alternatives to standard deviation and VaR in the last decade. Artzner et al. (1997) summarized the fundamental properties of a coherent risk measure. Based on these new risk measures, the Basel committee proposed two international regulatory capital requirements in 1988 and 2004, known as Basel I and Basel II (Basel Committee 1988 and Basel Committee 2004). Following suit, the European Commission has recently published the new, risk-based capital requirement framework known as Solvency II (European Commission 2008) for insurance companies. With the emergence of the new risk measures and risk-based capital requirement, Markowitz's risk-based portfolio optimization framework can be extended to robust forms.

The technique of portfolio optimization is not only useful for portfolio selection, it is also useful in areas like asset pricing, collateral allocation, and asset liability management. The basic idea is to apply risk measures in an optimal decision process.

Three categories of single-stage optimization approaches are typically seen for portfolios with stochastic nature: (1) reduced form—risk statistics for example standard deviation, risk weights are estimated separately from the optimization problem; (2) stochastic programming—risk is explicitly modeled in an optimization process; and (3) simulation-based optimization—optimization over a simulated sample space that turns stochastic programming into a deterministic programming problem. This paper presents several implementations of risk-based portfolio optimization using SAS Risk Management solutions that leverage SAS risk computation and optimization procedures. With the strong portfolio simulation power in SAS Risk Management solutions, all of the optimizations covered in this paper have an approach. "A Brief Review of Risk Measures" explains a few risk measures and portfolio optimization business cases. "Portfolio Optimization" introduces portfolio optimization that is used in portfolio selection. "Asset Liability Management with Stochastic Cash Flow" extends portfolio optimization to a cash-flow-based decision environment. "Optimal Credit Risk Mitigant Allocation" introduces an innovative optimal credit risk mitigant allocation algorithm in a credit risk portfolio under the new Basel II regulatory capital requirement framework. "Conclusion" concludes the paper.

A BRIEF REVIEW OF RISK MEASURES

For two portfolios—X and Y—Artzner et al. (1997) prescribed the following conditions for a risk measure *m* to be COHERENT:

- Subadditivity – $m(X+Y) \leq m(X) + m(Y)$: diversification is good for a portfolio.
- Homogeneity – $m(tX) = tm(X)$ *for any constant t*: no scale effect on a portfolio risk.
- Monotonicity – $m(X) \geq m(Y)$ *if X>Y almost surely:* risk is duely reflected in the risk measure.
- Translation invariance – $m(X+r) = m(X) + r$: risk-free asset does not contribute to overall risk.

Coherence of a risk measure is important for many reasons. Subadditivity and monotonicity together implies convexity, which is a necessary condition for the existence of a global optimum. A well-known result documented by Artzner et al. (1997) is that the widely accepted VaR measure introduced below is not guaranteed to be a coherent risk measure unless the underlying risk distribution belongs to an elliptic family.

The following risk measures are widely used in financial risk management:

- Standard Deviation: It is mostly used as a statistical measure to describe the volatility of random variables. In a nonparametric setting, the measure is defined as:

  $$\sigma = \sqrt{E[X - E(X)]^2},$$

  where *E* stands for the expectation of a random variable. In practice, people use the standard error from a sample space as an estimator of the standard deviation.

  The standard deviation captures the randomness of a normally distributed risk variable. It is also a coherence risk measure for most of the early work of the modern finance theory, including Markowitz's portfolio optimization, Sharpe (1964)'s capital asset pricing model (CAPM), and Black and Scholes (1973)'s option pricing formula. Only for elliptic-distributed underlying risk volatility, VaR , and expected shortfall are all equivalent because of the scalability of these distributions. For example, for normal distribution, VaR is simply a multiple of the standard deviation, where the multiplier is the corresponding percentile on a standard normal distribution.

- VaR: It has been a widely used measure in risk management since its birth. In simple business terms, it represents the maximum value or loss at a given confidence interval. Mathematically, it is defined as:

  $$VaR(\alpha) = \inf[x | P(X \geq x) \geq \alpha],$$

  where α is the given confidence interval. It is usually estimated in the same way as a percentile of a distribution for the given confidence interval α.

  VaR works for all distribution assumptions and provides intuitive business justification. As a result, it is used by many financial institutions despite its unwarranted coherence in general.

- Expected Shortfall (or Conditional VaR): It is a coherent risk measure. By definition, it is:

  $$ES(\alpha) = E[X | X \geq VaR(\alpha)]$$

  where $VaR(\alpha)$ is the VaR previously defined. For a fat tail distribution, expected shortfall is not guaranteed to exist. If it exists, it represents the expected value or loss beyond a given confidence interval. For a given sample space, expected shortfall is estimated as the mean of the points beyond and including VaR of a confidence interval α. This conditional mean calculation requires caution to remain convex.

SAS Risk Management solutions compute these risk measures and their variations using standard statistical algorithms.

An important application of these risk measures is to calibrate a financial institution's capital requirement. Such capital is a financial institution's own funds that are set aside to protect the institution from insolvency from big losses. This capital is oftentimes required to cover projected unexpected loss of a given confidence interval, which makes a connection to the risk measures. For example, the Basel II capital requirement for banks is based on 99% VaR for market risk (internal model approach) and 99.9% VaR for credit risk.

PORTFOLIO OPTIMIZATION

Portfolio optimization has wide applications in portfolio asset allocation and asset pricing. The traditional portfolio optimization is mean-variance optimization, which was originated by Markowitz (1952). The decision rule is to maximize the expected return of the portfolio, while keeping the return variance to a certain level. (Or to minimize the return variance for a given expected return.) The resulting portfolios for different desired variance levels make a so-called mean-variance efficient frontier. Mean-variance optimization led to the birth of CAPM, which was introduced by Sharpe (1964). In the following sections, mean-variance, mean-expected-shortfall (mean-ES), and mean-value-at-risk (mean-VaR) optimizations (supported in the SAS Risk Dimensions OPTIMIZATON statement) are explained.

MEAN-VARIANCE OPTIMIZATION

Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_M)$ be a vector of random returns of *M* assets in a portfolio.

Let $\boldsymbol{x} = (x_1, \ldots, x_M)'$ be a vector of the percentage of the portfolio invested in these assets.

The expected returns vector is $\overline{\boldsymbol{\theta}}$. The covariance matrix of returns on these *M* assets is $\boldsymbol{Q}$. The mean-variance optimization seeks to find an optimal vector $\boldsymbol{x}$ such that:

(1) $\boldsymbol{max_x x'\overline{\theta}}$

is subject to:

$$\boldsymbol{x}'\boldsymbol{Qx} \leq \delta^2$$

$$\sum_{i=1}^{M} x_i = 1$$

$$\boldsymbol{lb} \leq \boldsymbol{x} \leq \boldsymbol{ub},$$

where δ^2 is a preset tolerance of the variance of portfolio return, ***lb*** and ***ub*** are vectors of the lower and upper bounds of percentage weight on the assets in the portfolio. A user can also add linear business constraints (for example, weights on fixed-income assets in the portfolio retained at a certain level).

The optimal solutions to (1) from a range of δ^2 constraints result in a mean-variance efficient frontier. An example of the mean-variance efficient frontier created by SAS Risk Dimensions for a sample portfolio is illustrated in figure 1. It shows a portfolio of fixed incomes, derivatives, and equities based on a historical simulation over monthly interest rates, foreign exchange rates, and equity prices data from August 1998 to August 2001. The range of volatility constraint is (0, 0.03).

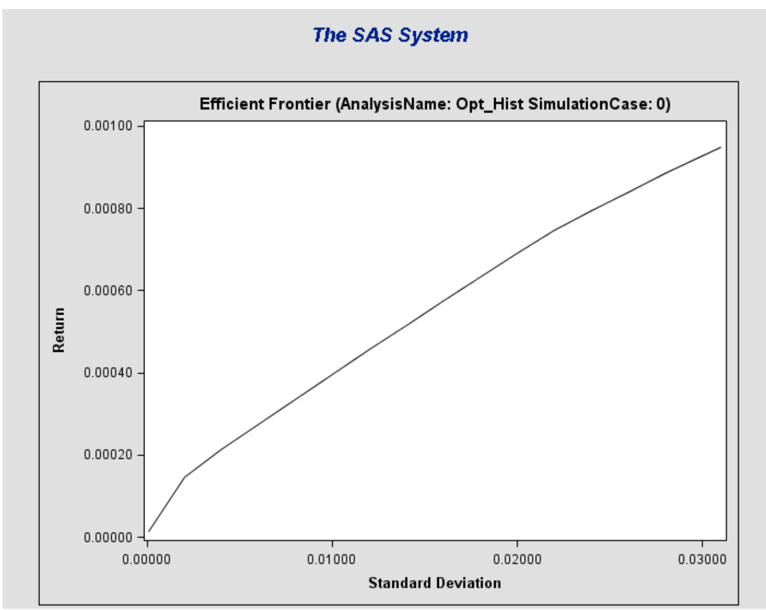

Figure 1. Mean-Variance Efficient Frontier of a Portfolio in an Historical Simulation

MEAN-ES AND MEAN-VAR OPTIMIZATIONS

Because of the limitation on volatility, mean-VaR optimization is becoming more popular in portfolio optimization. Both VaR and expected shortfalls are related to set of percentiles on a return distribution. Because expected shortfall is a convex risk measure, it is more amenable for global optimization than VaR and volatility. Let's assume a confidence interval α for the expected shortfall for the return distribution with density function $p(\cdot)$, which warrants a finite expected return of the portfolio. (The analytical presentation turns out to be unimportant for the optimization problem). Following the definition of expected shortfall and using the same notations as in (1), expected shortfall can be expressed as the conditional expectation of the portfolio return associated with choice of asset weights $\boldsymbol{x}$ can be expected as:

(2) $\varphi_\alpha(\boldsymbol{x},\gamma) = \gamma + \frac{1}{1-\alpha}\int[-\boldsymbol{x}'\boldsymbol{\theta} - \gamma]^+ \, p(\boldsymbol{\theta})d\boldsymbol{\theta},$

where γ is the smallest[1] solution to:

$\omega(\boldsymbol{x},\gamma) = \int I(-\boldsymbol{x}'\boldsymbol{\theta} \le \gamma)\, p(\boldsymbol{\theta})d\boldsymbol{\theta} = \alpha,\ [y]^+ = \begin{cases} y, & y > 0 \\ 0, & y \le 0 \end{cases}$ and $I(z) = \begin{cases} 1, & if\ z\ is\ true \\ 0, & otherwise \end{cases}.$

Rockafellar and Uryasev (2000) proved that (2) is a convex and continuously differentiable function. They show that in a sample space of *N*, simulated samples from the distribution of $\boldsymbol{\theta}$, can be further approximated by a convex (in $\boldsymbol{x},\gamma$) and piecewise linear (in γ) function:

$$\bar{\varphi}_\alpha(\boldsymbol{x},\gamma) = \gamma + \frac{1}{(1-\alpha)N}\sum_{j=1}^{N}\left[-\boldsymbol{x}'\boldsymbol{\theta}^j - \gamma\right]^+$$

Therefore, given a simulation with *N* scenarios, a mean-ES[2] optimization is equivalent to finding the solution for:

(3) $\boldsymbol{min}_{(\boldsymbol{x},\gamma)}\ \gamma + \frac{1}{(1-\alpha)N}\sum_{j=1}^{N} z^j$

which is subject to:

$$\boldsymbol{x}'\overline{\boldsymbol{\theta}} \ge \mu$$

$$z^j \ge 0$$

$$\boldsymbol{x}'\boldsymbol{\theta}^j + \gamma + z^j \ge 0,\ for\ j = 1, \dots, K$$

$$\sum_{i=1}^{M} x_i = 1$$

$$\boldsymbol{lb} \le \boldsymbol{x} \le \boldsymbol{ub},$$

where μ is the preset target of mean return, and z^j are the auxiliary variables that are derived from the simulated distribution of *N* simulated scenarios.

Similar to (1), linear business constraints can be added to (3) as well. Based on the above idea, SAS Risk Dimensions uses a scenario-based approach that converts a stochastic optimization into a deterministic optimization with simulated scenarios.

A mean-VaR optimization can be expressed as the following:

(4) $\boldsymbol{min}_{\boldsymbol{x}}[V_{[\alpha,N]}(-\boldsymbol{x}'\boldsymbol{\theta}^1, \dots, -\boldsymbol{x}'\boldsymbol{\theta}^N)]$

which is subject to:

$$\boldsymbol{x}'\overline{\boldsymbol{\theta}} \ge \mu$$

$$\sum_{i=1}^{M} x_i = 1$$

$$\boldsymbol{lb} \le \boldsymbol{x} \le \boldsymbol{ub}.$$

VaR is not a convex risk measure for most of the non-elliptic distributions, so (4) might not warrant a well-behaved optimization problem. However, Rockafellar and Uryasev (2000) have shown that from the presentation of the mean-ES problem (3), it can be seen that γ is a lower bound to the VaR at confidence interval α. Therefore, one can iteratively minimize (3) with respect to γ to approach the optimal solution of (4). On the other hand, a brute force nonlinear optimization using PROC NLP is also accommodated in SAS Risk Dimensions.

VaR and expected shortfall minimization might not always be a well-posed problem, even if the convexity is satisfied. Alexander et al. (2004) gave an example of an ill-posed optimization problem for a derivative portfolio where derivatives are evaluated with the delta-gamma approximation. The linear programming of an expected-shortfall-based approach becomes inefficient for a large portfolio in this case.

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[1] Equation $\omega(\boldsymbol{x},\gamma) = \alpha$ might contain more than one solution, although $\omega(\boldsymbol{x},\gamma)$ is a continuous and non-decreasing function.

[2] Krokhmal et al. (2002) showed that the mean-ES optimization in (3) generates the same efficient frontier as maximizing the mean return subject to a preset tolerance of expected shortfall using the same auxiliary variable representation. Both optimizations are supported in SAS Risk Dimensions.

More information about the implementation of the previous problems can be found in the documentation of the OPTIMIZATION statement in SAS Risk Dimensions.

ASSET LIABILITY MANAGEMENT WITH STOCHASTIC CASH FLOW

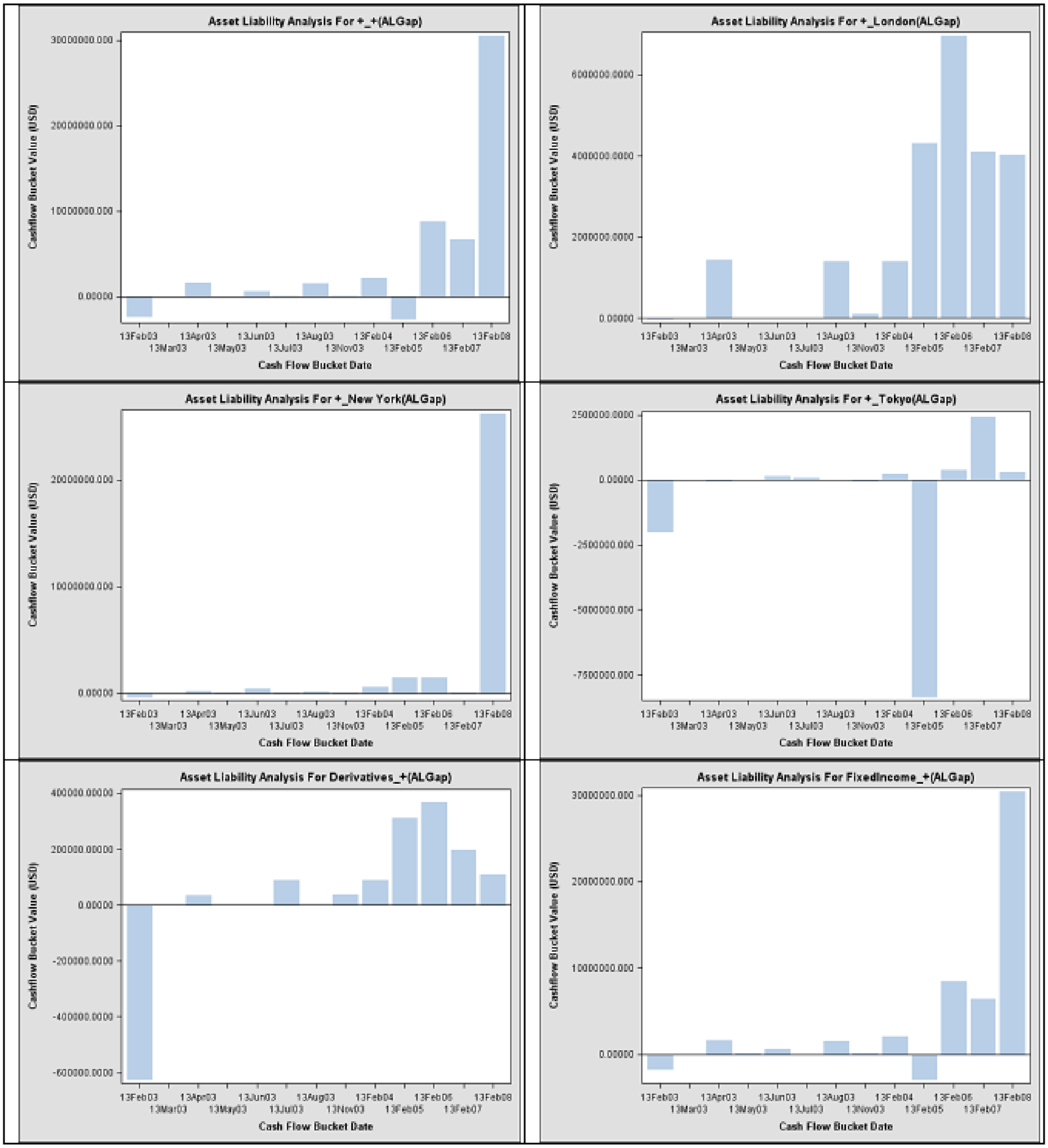

Figure 2. Sample Asset Liability Gap for a Range of Time Buckets

The previous optimization cases were based on the market value of the positions in a financial institution's portfolio. A lot of financial business is cash-flow-based. As market intermediator, a bank typically borrows from and lends to different finance sources and makes a profit from the process. A pension fund must produce income cash flow from its investments to support its payout commitment. Similarly, an insurance company relies on income from its assets and fees to cover its liabilities. For any institution with outstanding debts, enough cash flow from future business operations is critical for the institution to keep a good credit status. In these cases, decisions have to be made based on future cash flows generated from a portfolio of assets and liabilities. Ziemba (2003) has more details about the optimization for asset liability management.

Cash flows from positions in a portfolio can happen on different dates and be of different amounts. To simplify the cash flow analysis, financial institutions usually group cash flows into a set of predefined time buckets in the future. The choice of time buckets is specific to each institution's business strategy and operational life cycle. The bucketed cash flows are compared by asset and liability. Any significant gaps between the two might call for an alternate business strategy. There are many ways that gaps can be constructed. For example, one can compare the scheduled single or accumulated cash flow amounts in each bucket to look for a potential liquidity problem, or one can check the cash flows that are subject to rate adjustment (repricing) in future time periods, or one can compare durations and convexities (both are sensitivity measures) of the cash flows. For a portfolio of fixed incomes and derivatives across regions of London, New York, and Tokyo, figure 2 shows a portion of the cash flow gap report produced by SAS Risk Dimensions and SAS Risk Management for Banking.

SAS Risk Management for Banking enables the user to leverage the strong cash flow analysis of PROC RISK and the flexible optimization modeling of PROC OPTMODEL to accomplish decision objectives for asset and liability management. The process involves the following steps:

1. Simulate market states based on user-specified simulation methods (model-based, historical, or scenario).
2. Generate cash flows of user-defined types (maturity, repricing, interest income, and so on) for each simulated market state and collect them into buckets.
3. Analyze the bucketed cash flows by assets and liabilities (at each additional user-defined reporting hierarchy).
4. Execute optimization based on a selected objective function and constraints for the cash flows.

This process is driven by data in SAS Risk Dimensions.

In general, the following optimization problem yields an institution's asset allocation strategy to minimize the gap between its asset and liability by maturity.

$$\min_{x\in A}\sum_{t\in T} w_t \sum_{k=1}^{n} f[\sum_{i\in A,\, j\in L}(x_i C_{t,i}^k + C_{t,j}^k)] \tag{5}$$

Where:

$A =$ a set of all instruments of which the positions can be adjusted.
$L =$ a set of all fixed positions.
$n =$ the number of simulated scenarios.
$T =$ a set of time buckets.
$x_i =$ the number of shares of a particular asset instrument.
$C_{t,i}^k =$ the cash flow from either an asset or liability in time bucket *t* in a simulated scenario. Cash flows from liability carry negative signs.
$w_t =$ the weight of a time bucket.
$f(\cdot)$ the function that is determined by the decision rule.

This problem is subject to business constraints or risk (for example, VaR) constraints. Additional constraints might include $x_i \geq 0$ if short-selling assets are not allowed. For the performance of the optimization, usually x_i's are not restricted to integers. In most cases, the function $f(\cdot)$ is chosen as the least square or absolute function.

Equation (5) is a single-stage optimal portfolio allocation. A multi-stage problem would require dynamic portfolio rebalance over the life of the decision process. Multi-stage optimization might be possible with a dynamic trading and reinvestment feature that will soon be available in SAS Risk Dimensions.

A replicating portfolio has its roots in derivative pricing. The idea is to price a portfolio of illiquid assets with a set of liquid assets that have tangible market values. The liquid assets must be able to replicate the cash flow streams of the illiquid assets in all possible market scenarios. The recent birth of fair value rules has made the replicating portfolio approach more important. For example, the Solvency II framework for European insurance and reinsurance institutions has required the replicating portfolio technique to get the values of the insurance liabilities on the balance sheet. The objective function for the replicating portfolio can be similar to the minimization problem (5), where present values replace the straight values of all cash flows. That is, instead of $C_{t,i}^k$, the present value cash flow discounted in the corresponding simulated scenario $d_t^k C_{t,i}^k$ is applied. Risk constraints like variance, expected shortfall, and VaR can be applied to (5), which leads to a more risk-based replication. In the replicating portfolio application, assets have to be chosen to represent the unique features in the underlying liability structure, including optionality and payoff conditions. This topic is beyond the scope of this paper.

OPTIMAL CREDIT RISK MITIGANT ALLOCATION

Credit risk is one of the most significant risks a financial institution encounters every day. In general, credit risk is the risk of loss induced by the failure of a counterparty to fulfill the contracted obligation. Financial institutions have traditionally leveraged several tools such as guarantees, credit derivatives, collateral, or netting agreements to mitigate losses due to credit risk. The well-specified credit mitigation tools are often recognized in the capital requirement calculation. For example, Basel II has required the following:

- Credit exposures must be risk-weighted based on their credit qualities.
- If an eligible credit risk mitigant is present with an exposure, then the risk weight of the credit risk mitigant can be used as the risk weight of the portion of the exposure that is covered by the credit risk mitigant.
- The applicable value of the credit risk mitigant must be adjusted based on the nature of the mitigant.

Because an excessive capital level takes away an institution's business development opportunity, it is to the benefit of the institution to reduce its capital.

SAS introduced an optimal credit risk allocation method in its SAS Credit Risk Management for Banking solution using the generalized network algorithm. Before the method is described, consider the concept of a generalized network.

GENERALIZED NETWORK FOR AN EXPOSURE AND MITIGANT RELATION

Networks arise in numerous settings and in a variety of guises. Operations research has seen rapid advances in the methodology and application of network optimization models. A network representation provides a powerful visual and conceptual aid for portraying the relationships between the credit exposures and the credit risk mitigants. The generalized network optimization model in this context poses a special type of linear programming problem that can efficiently be solved with millions of exposures and mitigants that are in most banks.

Here are a few mathematical terms of the network models:

- Node is an entity in the network usually labeled as suppliers, demanders, or transition.
- Arc is the path from one entity to another.
- Directed network is a network that has only directed arcs.
- Arc cost is the cost associated with an arc when goods are moved from one end of the arc to the other.
- Arc multiplier is the discount factor associated with an arc when goods move through it.

Figure 3 is a directed network representation of an exposure and mitigant network. There are three exposures—E1, E2, and E3—and four credit risk mitigants—C1, C2, C3, and C4. The node NC supplies all of the unsecured portions of the exposures in the diagram. The credit risk mitigants act as supply nodes and exposures act as demand nodes. The flow costs are simply the risk weights associated with the mitigants. The arcs reflect the contractual relations between credit risk mitigants and exposures. In figure 3, exposure E1 is secured by C1 and C2, E2 is secured by C3 only, and E3 is secured by C3 and C4. Any amount flowed out of the NC node takes the risk weight of the correspondent exposure. The first number on each arc (exposure-mitigant relation) indicates the cost (risk weight) associated with the arc.

To adjust for the potential changes in the material value of the credit risk mitigants, additional adjustments are required based on the nature of the mitigants. For example, Basel II has required haircut discounts to financial collaterals and nettings. In addition, if the exposures and credit risk mitigants are in different currencies, further haircut discounts are required for the fluctuation of the foreign exchange rate dynamics. Because the discount might depend on both the exposure and the credit risk mitigant, it is arc dependent. Using the generalized network model, discounts can be added to the arc to represent the attrition in the goods. The second number on each arc indicates the multiplier associated with the arc.

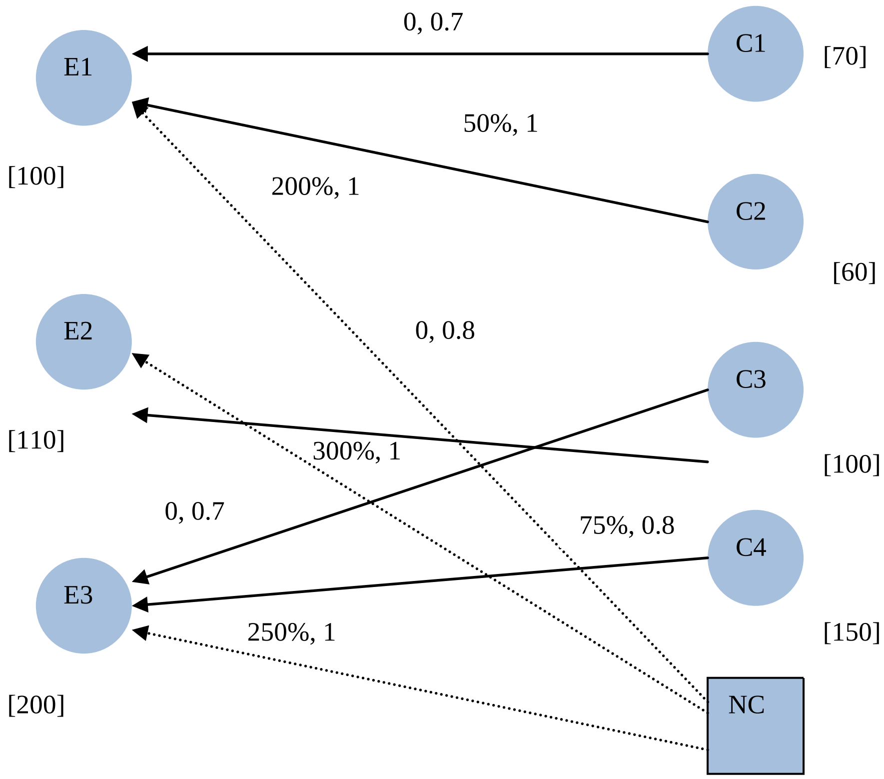

Figure 3. Example of an Exposure Mitigant Network

OBJECTIVE FUNCTION

The regulatory credit risk capital requirement proposed by Basel II can be summarized as the computation of the aggregated risk-weighted assets (which are a regulatory approximation of the VaR (previously introduced)):

(6) $$RWA = \sum_{i=1}^{n} EAD_i RW_i$$

RWA stands for risk-weighted asset, *EAD* stands for exposure at default, and *RW* is the risk weight. The total number of exposures is *n*. When credit risk mitigants are present with an exposure, the exposure can be decomposed into an unsecured portion and several secured portions that take the adjusted value of the credit risk mitigant. The secured portions take the risk weights that correspond to the mitigants. (Risk weight for a credit risk mitigant does not necessarily have the same interpretation as the risk weight for an exposure, but it has a similar effect.) The unsecured portion takes the risk weight of the exposure itself. The risk-weighted asset in equation (6) can now be written as:

(7) $$RWA = \sum_{i=1}^{n} \left(\sum_{j_i=1}^{m_i} SEC_{j_i} RW_{j_i} + USEC_i RW_i \right)$$

SEC is the secured portion and *USEC* is the unsecured portion. Each exposure can be secured by several credit risk mitigants. In general, multiple exposures covered by multiple credit risk mitigants should be considered in the methodology description.

With the introduction of the arc multiplier, the regulatory capital is related to the following equation:

(8) $$RWA = \sum_{i=1}^{n} \left(\sum_{j_i=1}^{m_i} SEC_{j_i} RW_{j_i} M_{j_i} + USEC_i RW_i \right)$$

The optimal credit risk mitigant allocation for the objective function in the Basel II specification becomes a minimum cost network flow problem with cost function (8).

The previous optimal credit risk mitigant allocation can be implemented using the CRM_ALLOCATION_METHOD="OPTIMAL" option in SAS Credit Risk Management.

AN EXAMPLE

Figure 3 further illustrates the problem and the proposed solution.

The Problem

A bank has three loan exposures—E1 through E3—and four credit risk mitigants—C1 through C4. Here is the detailed information.

1. E1 has EAD $100 and is risk-weighted 200%. E2 has EAD $110 and is risk-weighted 300%. E3 has EAD $200 and is risk-weighted 250%.
2. C1 and C3 are financial collaterals, C2 is a guarantee, and C4 is a receivable. C1 is worth $70. C2 is worth $60 and is risk-weighted 50%. C3 is worth $100. C4 is worth $150 and is risk-weighted 75%.
3. The directed arcs in figure 3 represent the possible coverage between the credit risk mitigants and exposures. They are typically specified by the business contracts and cannot be altered.

Financial collaterals are usually subject to haircut adjustments (collateral itself, and a currency mismatch adjustment, if necessary). Let's assume the haircut adjustments result in multipliers as (E1, C1)=0.7, (E2, C3)=0.8, and (E3, C3)=0.7. The guarantee has no adjustment, so the multiplier on (E1, C2)=1. The receivable is subject to an over-collateralization adjustment, which results in the multiplier (E3, C4)=0.8.

The regulatory capital requirement requires the risk-weighted asset to be calculated as equation (8). In addition, the bank needs to report the risk weight for the secured and unsecured portions of each exposure.

The Solution

In this simple example, there are a few ways that the bank can apply the credit risk mitigants to the exposures.

1. For E1, there are two possible ways to apply the mitigants.
   a. Apply C1 first and C2 second.
      The secured portion by C1 is $49 with 0 risk weight. The secured portion by C2 is $51 with 50% risk weight. The RWA for E1 is $25.50 (49*0+51*50%=25.5).
   b. Apply C2 first and C1 second.
      The secured portion by C2 is $60 with risk weight 50%. The secured portion by C1 is $40 with 0 risk weight. The RWA for E1 is $30 (60*50%+40*0=30).
2. E2 and E3 share C3, but E4 has sole claim to C4. There are two possible ways to apply the mitigants.
   a. E2 is secured by C3 and E3 is secured by C4.
      The secured portion of E2 by C3 is $80 with 0 risk weight. The unsecured portion is $30 with 300% risk weight. The RWA for E2 is $90 (80*0+30*300%=90).
      The secured portion of E3 by C4 is $120 with 75% risk weight. The unsecured portion is $80 with 250% risk weight. The RWA for E3 is $290 (120*75%+80*250%=290).
      The total RWA for E2 and E3 is $380.
   b. E3 is secured by C3 and C4. E2 is completely unsecured.
      The secured portion of E3 by C3 is $70 with risk weight 0. The secured portion of E3 by C4 is $120 with risk weight 75%, which leaves $10 unsecured. The RWA for E3 is $185 (70*0+120*75%+10*250%=185). The RWA for E2 unsecured is $330 (110*300%=330).
      The total RWA for E2 and E3 is $515.

Overall, there are four possible ways to apply the credit risk mitigants in this example. The optimal solution is the 1(a) and 2(a) combination, where the total RWA is $405.5. The worst solution is the 1(b) and 2(b) combination, where the total RWA is $545. For a bank with millions of exposures and credit risk mitigants, each worth hundreds of thousands of dollars, the differences in the RWA because of mitigant allocations can be drastic. Having an optimal solution is not trivial. SAS Credit Risk Management uses PROC OPTLP to solve this problem.

Because exposures and credit risk mitigants only form a closed network when they are related on the contractual base, the minimal cost network optimization technique can be applied to each independent sub-network without sacrificing the global optimum. In the above example, (E1, C1, C2) and (E2, E3, C3, C4) are two disjointed networks.

A network partition algorithm, based on the contractual relations between exposures and mitigants, is used in SAS Credit Risk Management to recognize all of the sub-networks.

A point worth making is that the optimal allocation might not always be allowed if a business contract clearly states how credit risk mitigants should be applied. SAS Credit Risk Management provides users with an option to honor the contractual allocation, instead of the optimal allocation. Even if this is the case, the optimal allocation is still useful for decision makers to get more insightful information on how risk can be reduced in an alternative way.

The generalized network approach to the optimal credit risk mitigant allocation provides an intuitive and efficient algorithm to allocate exposures and credit risk mitigants to minimize credit risk. The objective function can be extended beyond the Basel II capital requirement. When the problem extends with simulated scenarios, the variance, VaR, and expected shortfall optimization techniques can be applied.

CONCLUSION

Risk-based portfolio optimization supports better financial decision making on a risk-adjusted basis. In a fluctuating financial market, a risk-based decision process becomes more essential for a financial institution. This paper presents several portfolio optimization methods in SAS Risk Management solutions that use risk modeling, analysis, and optimization. These methods have many applications beyond financial portfolios. In a case where a decision can be modeled with uncertainty (for example, a real option problem), a risk-based portfolio optimization can help reach an intuitive result.

SAS Risk Management solutions strive to provide cutting edge solutions for decision making in a risk environment.

REFERENCES

Alexander, S., T. F. Coleman, and Y. Li. 2004, "Minimizing VaR and CvaR for a Portfolio of Derivatives." Working Paper, Cornell University, 2004.

Artzner, P., F. Delbaen, J. M. Eber, and D. Heath. 1997. "Coherent Measures of Risk." *Mathematical Finance* 9:203-228.

Black, F., and M. S. Scholes. 1973. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81(3):637-654.

Krokhmal, P., J. Palmquist, and S. Uryasev. 2002. "Portfolio Optimization with Conditional Value-at-Risk Objective and Constraints." *Journal of Risk* 4(2):43-68.

Markowitz, H. M. 1952. "Portfolio Selection." Journal of Finance 7:77-99.

Rockafellar, R.T., and S. Uryasev. 2000. "Optimization of Conditional Value-at-Risk." *Journal of Risk* 2(3):21-39.

Sharpe, W. F. 1964. "Capital asset prices: A theory of market equilibrium under conditions of risk." *Journal of Finance* 19:425-442.

Ziemba, W.T. 2003. *The Stochastic Programming Approach to Asset, Liability, and Wealth Management.* Research Foundation of CFA Institute.

RESOURCES

Basel Committee. 1988. *International convergence of capital measurement and capital standards.* Available at http://www.bis.org/publ/bcbs04a.htm.

Basel Committee. 2004. *Basel II: International Convergence of Capital Measurement and Capital Standards: a Revised Framework*. Available at http://www.bis.org/publ/bcbs107.htm.

Commission of the European Communities. 2008. *Amended Proposal for a Directive of the European Parliament and of the Council on the taking-up and pursuit of the business of Insurance and Reinsurance* - (Solvency II). Available at http://ec.europa.eu/internal_market/insurance/solvency/index_en.htm#proposal.

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